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THE LOGARITHM AS A DIRECT FUNCTION

By J. W. Bradshaw

WITH AN INTRODUCTION BY

W. F. Osgood

The student of mathematics and physics meets logarithms for the first time at an early stage. He is told that "the logarithm of a number is the exponent of the power to which a certain number, taken as the base, must be raised in order to equal the given number." The definition is purely formal. Probably the beginner has never seen a proof for the existence even of the fifth root of 2, — the square, cube, and fourth roots can be constructed geometrically, — and if he has, it is not likely that it has meant anything to him. He tacitly assumes that every positive number has a positive qth root, q being any positive integer. It is then an easy step to any rational power, and the irrational powers are thought of as limiting cases, the principle being that, whenever one wants a limit in mathematics, the limit exists.

Now all of these assumptions have been justified by rigorous ϵ-proofs in well known treatises on modern analysis.* But the general student of mathematics and physics does not read these proofs, for they are uninteresting to him; and thus the great majority of students of the Calculus never see a proof that there is such a thing as a logarithm.

It is possible, however, to supply a proof by means of elementary calculus, inclusive of the general theorems about continuous functions with which all students are familiar. How simple the analysis is appears from a casual glance at the following pages, in which Mr. Bradshaw has carried through all the details of a rigorous development of the essential properties of the Logarithm, and, as it appears here, of its *Inverse*, the Exponential Function, a^x. It is hoped that this presentation may prove attractive to students who have finished a thorough course in elementary calculus. W. F. O.

* Stolz, *Allgemeine Arithmetik*, vol. 1, ch. 8; new edition by Stolz and Gmeiner, under the title *Theoretische Arithmetik*, vol. 1, ch. 8. J. Tannery, *Fonctions d'une variable*, ch. 3. E. V. Huntington, *Grundoperationen an absoluten und komplexen Grössen*, Strassburg dissertation, 1901.

1. Definition of the Logarithm. The logarithm is commonly defined as the inverse of the exponential. It may, however, be defined as a direct function and treated quite independently. The exponential function will then appear as the inverse of the logarithm.

The logarithm of x shall here be defined as the definite integral

$$\int_1^x \frac{dx}{x}$$

for all values of the variable in the interval $0 < x < \infty$. We shall denote it by $\phi(x)$.

If we draw the curve

$$y = \frac{1}{x},$$

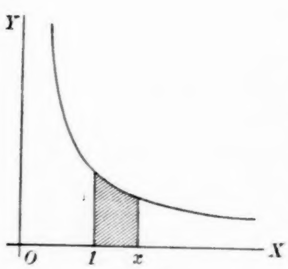

FIG. 1.

$\phi(x)$ is the area bounded by this curve, the x-axis, and two ordinates, one of which is fixed, $x = 1$, the other variable. The area is positive if the variable ordinate is to the right of the fixed ordinate, negative if to the left. If the two ordinates coincide, the area is zero, so that

$$\phi(1) = 0.$$

Since the curve $y = 1/x$ does not cut the x-axis, the area always increases as the variable ordinate moves to the right. $\phi(x)$ is therefore a single valued function of x, which everywhere increases with x.

The following theorems will serve to disclose the chief characteristics of the function.

THEOREM I. *As x increases indefinitely, $\phi(x)$ becomes positively infinite:—*

$$\phi(\infty) = \int_1^\infty \frac{dx}{x} = +\infty \cdot$$

As x approaches zero, $\phi(x)$ becomes negatively infinite:—

$$\phi(0) = \int_1^0 \frac{dx}{x} = -\infty \cdot$$

The proof of this theorem is ordinarily based in the Calculus on the

property of $\log x$ that $\log \infty = \infty$. Here it is necessary, of course, to give a direct proof.

Allow x to take any two values, m and l, where $m > l \geqq 1$, and note the difference in the corresponding values of $\phi(x)$.

$$\phi(m) - \phi(l) = \int_1^m \frac{dx}{x} - \int_1^l \frac{dx}{x} = \int_l^m \frac{dx}{x}.$$

By the mean value theorem (v. Picard, *Traité d'analyse*, vol. 1, p. 7),

$$\int_l^m \frac{dx}{x} = (m - l)\,\frac{1}{\mu}, \qquad l < \mu < m.$$

If we put $m = 2l$,

$$\phi(2l) - \phi(l) = l\,\frac{1}{\mu} > l/2l = 1/2.$$

Hence

$$\phi(2l) > \phi(l) + 1/2.$$

Since $\phi(1) = 0$, we have

$$\phi(2) > 1/2, \quad \phi(4) > 1, \ldots, \phi(2^n) > n/2.$$

If we have given any positive constant G, however large, and if we choose $n > 2G$, then for all values of $x > 2^n$, $\phi(x) > G$. Therefore $\phi(x)$ becomes positively infinite as x increases indefinitely, and thus the first part of the theorem is proved.

To prove the second part of the theorem, put $x = 1/x'$; then

$$\phi(x) = \int_1^x \frac{dx}{x} = -\int_1^{x'} \frac{dx}{x} = -\phi(x').$$

As x approaches zero, x' increases indefinitely and $\phi(x')$ becomes positively infinite; hence $\phi(x)$ becomes negatively infinite.

Theorem II. *The function $\phi(x)$ is continuous in the interval $0 < x < \infty$. It has, at each point of this interval, a derivative which is also continuous and which is given by the formula:*

$$\phi'(x) = \frac{1}{x}.$$

This theorem follows at once from the theorem of the Integral Calculus that the function

$$F(x) = \int_{x_0}^{x} f(x)\,dx,$$

where $f(x)$ is continuous, is a continuous function having $f(x)$ as its derivative. The proof is given here for completeness.

We wish to prove first that, if x_0 is any positive number,

$$\lim_{x = x_0} \phi(x) = \phi(x_0).$$

Let $x_0 + \Delta x$ be any positive value for x near x_0. Then

$$\phi(x_0 + \Delta x) - \phi(x_0) = \int_1^{x_0+\Delta x} \frac{dx}{x} - \int_1^{x_0} \frac{dx}{x}$$

$$= \int_{x_0}^{x_0+\Delta x} \frac{dx}{x} = \Delta x \, \frac{1}{x_0 + \theta\Delta x}, \qquad 0 < \theta < 1.$$

Since $x_0 + \theta\Delta x$ approaches x_0 as its limit, the limit of the second member is 0, and hence the limit of the first member is 0. Therefore

$$\lim_{\Delta x = 0} \phi(x_0 + \Delta x) = \phi(x_0),$$

and $\phi(x)$ is continuous.

Next form the difference quotient at the point x_0:

$$\frac{\phi(x_0 + \Delta x) - \phi(x_0)}{\Delta x} = \frac{1}{x_0 + \theta\Delta x}.$$

Then

$$\lim_{\Delta x = 0} \frac{\phi(x_0 + \Delta x) - \phi(x_0)}{\Delta x} = \phi'(x_0) = \frac{1}{x_0}.$$

Hence $\phi(x)$ has a derivative, $1/x$, and this is continuous throughout the interval $0 < x < \infty$.

To sum up, we have shown that $\phi(x)$ is a function of x which is single-valued and continuous in the interval $0 < x < \infty$, which always increases with x, and which has at each point of the interval a positive continuous derivative,

$\phi'(x) = 1/x$; that $\phi(1) = 0$, $\phi(0) = -\infty$, and $\phi(\infty) = \infty$. The graph of the function is therefore as shown in the figure.

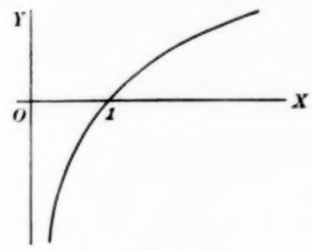

Fig. 2.

2. The Inverse Function. Theorem. *The equation*

$$y = \phi(x),$$

x lying in the interval $0 < x < \infty$, defines x as a function of y:

$$x = \psi(y)$$

which is single valued and continuous in the interval $-\infty < y < \infty$, increases with y, and has, at each point of the interval, a continuous positive derivative.

The function $\psi(y)$ will turn out to be e^y. We wish to prove first that, any value y_0 being given, there exists one and only one value x_0 of x satisfying the equation*

$$y_0 = \phi(x). \tag{1}$$

Geometrically this means that the line $y = y_0$ cuts the curve $y = \phi(x)$ in one and only one point. An examination of the figure shows at once that this is the case, and as an intuitive proof this is sufficient. An analytic proof may be given by means of the following well known theorem of continuous functions.

As x varies continuously from a to b, any function $f(x)$, continuous in the interval $a \leqq x \leqq b$, passes through all values included between $f(a)$ and $f(b)$.

Since $\phi(0) = -\infty$ and $\phi(\infty) = +\infty$, $\phi(x)$ must have taken on the value y_0 for some positive value x_0 of x. Equation (1) has therefore one solution, x_0. It has, moreover, only one solution. Suppose there were a second positive solution x_1. This value cannot be greater than x_0, for, if it were, $\phi(x_1)$ would be greater than $\phi(x_0)$, since $\phi(x)$ always increases with x. Similarly x_1 cannot be less than x_0. Hence x is a single valued function of y in the interval $-\infty < y < \infty$. Call it $\psi(y)$.

If y_0, y_1 are any two values of y, and $y_0 < y_1$, then evidently $\psi(y_0) < \psi(y_1)$.

To show that $x = \psi(y)$ is a continuous function of y, take an arbitrary

* This theorem is proven in advanced treatises on the Differential Calculus; *cf.* Stolz, *Differential- und Integralrechnung*, vol. 1, p. 38.

value y_0 of y and an auxiliary interval

$$P: \quad 0 < a < x < b$$

so as to include the point $x_0 = \psi(y_0)$.

Let $\phi(a) = A$ and $\phi(b) = B$. Then $A < y_0 < B$. Let $y_0 + \Delta y$ be any second value of y lying between A and B, and let $x_0 + \Delta x$ be the corresponding value of x; it must be in the interval P. We wish to show that when Δy approaches 0, Δx also approaches 0.

$$y_0 = \phi(x_0), \qquad y_0 + \Delta y = \phi(x_0 + \Delta x),$$

$$\begin{aligned} \Delta y &= \phi(x_0 + \Delta x) - \phi(x_0), \\ &= \Delta x\, \phi'(x_0 + \theta\Delta x), \qquad 0 < \theta < 1, \\ &= \Delta x \frac{1}{x_0 + \theta \Delta x}. \end{aligned}$$

But $x_0 + \theta\Delta x$ lies in the interval P, hence

$$|\Delta y| > |\Delta x| \frac{1}{b}, \qquad |\Delta x| < b|\Delta y|.$$

Therefore Δx approaches 0 when Δy does, and $\psi(y)$ is continuous at the point y_0, which was any point.

Furthermore $\psi(y)$ has a continuous derivative. For the difference quotient is given by the formula

$$\frac{\Delta x}{\Delta y} = x_0 + \theta \Delta x .$$

Since Δx approaches 0 when Δy does,

$$\lim_{\Delta y = 0} \frac{\Delta x}{\Delta y} = x_0 .$$

Hence $\psi(y)$ has a derivative

$$\psi'(y) = \psi(y),$$

and this derivative is continuous.

3. The Fundamental Property of Logarithms. We will now prove the fundamental property of logarithms, that the sum of the logarithms of two positive numbers is equal to the logarithm of their product. We have to prove that at each point of the region

$$R: \begin{cases} 0 < x < \infty \\ 0 < y < \infty \end{cases}$$

the functional equation

$$\phi(x) + \phi(y) = \phi(xy)$$

is satisfied. To do this, write the left hand side of the equation in the form

$$\int_1^x \frac{dt}{t} + \int_1^y \frac{dt}{t}$$

and then replace the variable of integration in the second integral by t', where*

$$t' = xt.$$

We have, then,

$$\int_1^y \frac{dt}{t} = \int_x^{xy} \frac{dt'}{t'}.$$

Hence, dropping the accent in this last integral and substituting above, we obtain

$$\int_1^x \frac{dt}{t} + \int_x^{xy} \frac{dt}{t} = \int_1^{xy} \frac{dt}{t},$$

or $\phi(xy)$, as the value of $\phi(x) + \phi(y)$, and this is the result which we set out to establish.

4. A Second Property of the Function $\phi(x)$. Existence of Roots. From the law

$$\phi(x) + \phi(y) = \phi(xy) \tag{2}$$

we will now deduce another property of the function ϕ, namely,

$$\phi(x^a) = a\phi(x), \tag{3}$$

where a is any rational number. Incidentally a proof is obtained of the existence of a positive qth root of any positive number a and hence of a positive value for every commensurable power of a : $a^{p/q}$.

Equation (2) gives us at once the following facts with regard to the function ϕ:

$$\phi(1/x) + \phi(x) = \phi(1) = 0,$$

$$\therefore \phi(1/x) = -\phi(x). \tag{4}$$

* This method has been employed in the study of the logarithmic function for complex values of the argument by Burkhardt, *Analytische Funktionen*, p. 162.

$$\phi(x) + \phi(x) + \cdots \text{ to } n \text{ terms} = \phi(x \cdot x \cdot \cdots \text{ to } n \text{ factors}),$$

$$\therefore\ n\phi(x) = \phi(x^n), \tag{5}$$

n being a positive integer.

Assuming, as we have in the last equation, the definition of a^n when n is a positive integer, we can prove the

THEOREM. *Every positive number a has one, and only one, positive qth root, q being a positive integer.*

Suppose a has a positive qth root, x; then

$$a = x^q,$$

$$\phi(a) = \phi(x^q) = q\phi(x).$$

Let $\phi(a) = b$, and we have

$$x = \psi\left(\frac{b}{q}\right). \tag{6}$$

This is a necessary relation for any positive qth root of a, and it shows, too, that if a has a positive qth root it can have but one, since ψ is a single valued function. Further, we know there is a positive number x satisfying (6), and, retracing our steps, the qth power of x is a. Therefore the number x determined by (6) is a positive qth root of a, and the only one.

If we define $a^{p/q}$ as $(\sqrt[q]{a})^p$* and $a^{-p/q}$ as $\dfrac{1}{(\sqrt[q]{a})^p}$, the theorem shows that each of these expressions has one and only one positive value. If further a^0 is defined as 1, a^α, assumed positive, is defined and single valued for all rational values of α.

If in (5) we put $y = x^n$, we have

$$n\phi(y^{\frac{1}{n}}) = \phi(y),$$

$$\therefore\ \phi(y^{\frac{1}{n}}) = \frac{1}{n}\phi(y). \tag{7}$$

$$\phi(y^{\frac{1}{n}}) + \phi(y^{\frac{1}{n}}) + \cdots \text{ to } m \text{ terms} = \frac{m}{n}\phi(y),$$

and also

$$= \phi[(y^{\frac{1}{n}})^m],$$

$$\therefore\ \phi(y^{\frac{m}{n}}) = \frac{m}{n}\phi(y). \tag{8}$$

* It will appear later that $(\sqrt[q]{a})^p = \sqrt[q]{a^p}$.

From (9)

$$\phi\left(\frac{1}{y^{\frac{m}{n}}}\right) = -\phi(y^{\frac{m}{n}}),$$

$$\therefore \phi(y^{-\frac{m}{n}}) = -\frac{m}{n}\phi(y). \tag{9}$$

Finally, since

$$\phi(x^0) = \phi(1) = 0, \tag{10}$$

we have, for all commensurable numbers a, the relation

$$\phi(x^a) = a\phi(x). \tag{3}$$

From this law may be deduced at once as corollaries the following theorems :

1°. $$\sqrt[q]{a^p} = (\sqrt[q]{a})^p.$$

2°. $$\sqrt[mq]{a^{mp}} = \sqrt[q]{a^p}.$$

3°. $$\sqrt[q]{\sqrt[m]{a}} = \sqrt[mq]{a}.$$

4°. $$\sqrt[q]{a^p}.\sqrt[q]{b^p} = \sqrt[q]{(ab)^p}.$$

5. Definition of the Exponential Function. Give to x any fixed positive value a and replace a by x; let $\phi(a) = b$; (3) then becomes

$$\phi(a^x) = xb, \tag{11}$$

or

$$a^x = \psi(xb), \tag{12}$$

a relation which holds for all rational values of x. We shall define a^x for irrational values of x by this relation. It follows that a^x is a single valued, positive, continuous function of x, and that it has a continuous derivative

$$\frac{da^x}{dx} = \frac{d}{dx}\psi(xb) = b\psi'(xb) = b\psi(xb) = a^x\phi(a).$$

The laws of indices may be derived very simply from (11). Let x and y be any real numbers and a and b positive numbers.

1) $$a^x a^y = a^{x+y}. \tag{13}$$

For

$$\phi(a^x a^y) = \phi(a^x) + \phi(a^y)$$
$$= x\phi(a) + y\phi(a) = (x+y)\phi(a) = \phi(a^{x+y}).$$
$$\therefore a^x a^y = a^{x+y}.$$

Similarly it is shown that

$$2)\qquad (a^x)^y = a^{xy},$$
$$3)\qquad a^x b^x = (ab)^x. \tag{14}$$

6. Two Functional Relations for the Function $\psi(x)$. If we define the number e by the equation

$$\phi(e) = 1,$$

equation (12) becomes

$$e^x = \psi(x).$$

Using the special value e for a in equation (13), we have the first functional relation

$$\psi(x) \cdot \psi(y) = \psi(x + y). \tag{15}$$

Substituting e for a in (14) we have a second functional relation

$$\psi(xy) = [\psi(x)]^y. \tag{16}$$

7. The Sufficiency of the Functional Relations. It turns out that the functional relations (2), (15) are sufficient for the determination of the functions, as is stated more precisely in the following theorems.

THEOREM. $\phi(x)$ *is the only function of x, single valued and continuous for all positive values of x, which satisfies the functional relation*

$$\Phi(x) + \Phi(y) = \Phi(xy) \tag{2}$$

and takes on the value 1 when $x = e$.

Suppose there were a second function, $\Phi(x)$, meeting all these requirements. Equation (2) shows that $\Phi(1) = 0$. Then by the reasoning by which (11) was derived from (2),

$$\Phi(a^x) = x\Phi(a),$$

when x is rational and hence, because of the continuity of the function Φ, for all values of x; or e being substituted for a,

$$\Phi(e^x) = x\,\Phi(e) = x = \phi(e^x).$$

Since e^x takes on all positive values, $\Phi(x)$ and $\phi(x)$ are equal for all positive values of x.

The function $\phi(x)$ thus turns out to be the natural logarithm of x.

A similar theorem holds regarding the function $\psi(x)$.

THEOREM. *$\psi(x)$ is the only single valued, positive, continuous function of x which satisfies the relation*

$$\Psi(x)\cdot\Psi(y)=\Psi(x+y) \tag{15}$$

and which has the value e when $x=1$.

By a course of reasoning similar to that by which (11) was derived from (2), it may be shown that any function Ψ which satisfies the conditions of the theorem satisfies also the relation

$$\Psi(xy)=[\Psi(y)]^x,$$

x being any rational number, and hence, because of the continuity, any number whatever. Put $y=1$ and we have

$$\Psi(x)=e^x=\psi(x).$$

8. The Logarithm to any Base. The function $\phi(x)$ which we have been considering is the logarithm to the base e. The logarithm to any other base, a, may be defined thus:

$$\log_a x=\frac{\phi(x)}{\phi(a)}.$$

Any positive number except 1 can be used as base. It will be seen that this function possesses the chief properties of $\phi(x)$; $\log_a x$ is a single valued, continuous function for all positive values of x, has a continuous derivative $\dfrac{1}{x\phi(a)}$, and satisfies the fundamental law

$$\log_a x+\log_a y=\log_a xy.$$

The following relations are often useful.

1°. If $$a^x=b, \qquad a\neq 1,$$

then $$\phi(a^x)=x\phi(a)=\phi(b),$$

$$x=\frac{\phi(b)}{\phi(a)},$$

and therefore $$b=a^{\log_a b}.$$

2°. If $$b=a^x=c^{xy}, \qquad a, c\neq 1,$$

then $$y=\frac{\phi(a)}{\phi(c)},\quad xy=\frac{\phi(b)}{\phi(c)},$$

whence

$$x = \frac{\phi(b)}{\phi(c)} \div \frac{\phi(a)}{\phi(c)} = \frac{\log_c b}{\log_c a},$$

and therefore

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

HARVARD UNIVERSITY,
JUNE, 1902.

ON POSITIVE QUADRATIC FORMS

BY PAUL SAUREL

THE necessary and sufficient conditions that a homogeneous quadratic function of n variables be constantly positive or constantly negative are well known. A very simple demonstration of the necessity of these conditions has been given by Gibbs in his great memoir On the Equilibrium of Heterogeneous Substances.* This demonstration, however, has not received the attention which it deserves, perhaps because its simplicity is somewhat disguised by the physical terms employed. In the present note we shall reproduce Gibbs's demonstration and we shall complete it by showing that certain of the conditions thus obtained are sufficient.

Let us consider the quadratic function ϕ defined by the equation

$$\phi = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} x_i x_k, \tag{1}$$

in which

$$a_{ik} = a_{ki}, \tag{2}$$

and let us write

$$f_i = \sum_{k=1}^{n} a_{ik} x_k. \tag{3}$$

From (3) we get

$$df_i = \sum_{k=1}^{n} a_{ik} dx_k. \tag{4}$$

* *Transactions of the Connecticut Academy of Arts and Sciences*, vol. 3, part 1, page 166 (1876).

and from this in turn we get

$$\sum_{i=1}^{n} df_i dx_i = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} dx_i dx_k. \tag{5}$$

From equations (1) and (5) it follows, when we remember that the differentials of the independent variables are entirely arbitrary, that the necessary and sufficient condition that ϕ be always positive is that

$$\sum_{i=1}^{n} df_i dx_i > 0. \tag{6}$$

By giving to the differentials in this inequality special values, we can deduce from it a great variety of necessary conditions. Certain of these, which we shall ultimately prove to be sufficient conditions also, we now deduce.

Setting $dx_1 \neq 0$, $dx_2 = dx_3 = \cdots\cdot = dx_n = 0$, and dividing (6) by the positive quantity dx_1^2, we have as a first necessary condition

$$\frac{\partial f_1}{\partial x_1} > 0.$$

Let us now introduce in place of the independent variables $x_1, x_2, \cdots, x_n$ the new independent variables $f_1, x_2, \ldots, x_n$. This will be possible if $\partial f_1/\partial x_1 \neq 0$, and therefore, in particular, if the necessary condition just obtained is fulfilled. Now set $dx_2 \neq 0$, $df_1 = dx_3 = \cdots\cdot = dx_n = 0$ and divide (6) by dx_2^2 getting as a second necessary condition

$$\frac{\partial f_2}{\partial x_2} > 0,$$

where, however, we must remember in forming the partial derivative that the independent variables are now $f_1, x_2, \ldots, x_n$.

Next pass from the independent variables just used to the independent variables $f_1, f_2, x_3, \ldots, x_n$ — a change of variable which can be made if the last written inequality is fulfilled — and proceed as before. In this way we get the following set of necessary conditions:

$$\left(\frac{\partial f_1}{\partial x_1}\right)_{x_1, x_2, x_3, \ldots, x_n} > 0,$$

$$\left(\frac{\partial f_2}{\partial x_2}\right)_{f_1, x_2, x_3, \ldots, x_n} > 0,$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

$$\left(\frac{\partial f_{n-1}}{\partial x_{n-1}}\right)_{f_1, \ldots, f_{n-2}, x_{n-1}, x_n} > 0,$$

$$\left(\frac{\partial f_n}{\partial x_n}\right)_{f_1, \ldots, f_{n-2}, f_{n-1}, x_n} > 0,$$

in which the subscripts indicate the quantities which are regarded as the independent variables.

The quantities which appear in inequalities (7) can be put into another form. Denote by Δ_n the determinant

$$\begin{vmatrix} a_{11} & a_{12} & . & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & . & a_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & . & a_{nn} \end{vmatrix},$$

and by $\Delta_{n-1}, \Delta_{n-2}, \ldots, \Delta_1$ the determinants obtained by erasing the last row and the last column of $\Delta_n, \Delta_{n-1}, \ldots, \Delta_2$. If we make use of equations (3), we obtain without difficulty the following equations:

$$\left(\frac{\partial f_1}{\partial x_1}\right)_{x_1, x_2, x_3, \ldots, x_n} = \Delta_1,$$

$$\left(\frac{\partial f_2}{\partial x_2}\right)_{f_1, x_2, x_3, \ldots, x_n} = \frac{\Delta_2}{\Delta_1},$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

$$\left(\frac{\partial f_{n-1}}{\partial x_{n-1}}\right)_{f_1, \ldots, f_{n-2}, x_{n-1}, x_n} = \frac{\Delta_{n-1}}{\Delta_{n-2}},$$

$$\left(\frac{\partial f_n}{\partial x_n}\right)_{f_1, \ldots, f_{n-2}, f_{n-1}, x_n} = \frac{\Delta_n}{\Delta_{n-1}}.$$

Conditions (7) may accordingly be replaced by the conditions:

$$\Delta_1 > 0, \quad \Delta_2 > 0, \ldots, \quad \Delta_n > 0. \qquad (9)$$

The demonstration by which we have established the necessity of these conditions is that of Gibbs.

We shall now show that conditions (7) are not only necessary but are also sufficient. For this purpose it will be enough to show that if conditions of the form (7) are sufficient when $n-1$ variables are involved, they are also sufficient when n variables are involved.

By referring to equations (1) and (3) it is obvious that we can write

$$\phi = \sum_{i=1}^{n} f_i x_i. \tag{10}$$

We can throw this equation into the form

$$\phi = \frac{f_1^2}{a_{11}} + \sum_{i=2}^{n} f_i' x_i, \tag{11}$$

where

$$f_i' = \frac{a_{11} f_i - a_{1i} f_1}{a_{11}}. \tag{12}$$

It should be noticed that f_i' is independent of x_1.

If conditions (7) are sufficient when $n-1$ variables are involved it follows from equation (11) that ϕ will certainly be positive if the following conditions hold:

$$\begin{gathered} a_{11} > 0, \\ \left(\frac{\partial f_2'}{\partial x_2}\right)_{x_3, x_3, x_4, \ldots, x_n} > 0, \\ \left(\frac{\partial f_3'}{\partial x_3}\right)_{f'_2, x_3, x_4, \ldots, x_n} > 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \left(\frac{\partial f_n'}{\partial x_n}\right)_{f'_2, f'_3, \ldots, f'_{n-1}, x_n} > 0. \end{gathered} \tag{13}$$

Since f_i' is independent of x_1, we may give to dx_1 any convenient value. It will therefore be allowable to suppose that in each of the differential coefficients in (13) dx_1 has been so taken that

$$df_1 = 0. \tag{14}$$

But, in that case, equation (12) shows that

$$df_i' = df_i. \tag{15}$$

If we make use of equations (14) and (15), conditions (13) reduce at once to conditions (7).

Thus, if conditions (7) be sufficient conditions in the case of $n-1$ variables they are also sufficient conditions in the case of n variables. As these conditions are obviously sufficient in the case of one variable, they are sufficient in general.

If we wish to obtain the necessary and sufficient conditions that ϕ be constantly negative we must reverse the sign of inequality in each of conditions (7). The signs of the determinants in (9) will then be alternately negative and positive.

NEW YORK, JULY, 1902.

MULTIPLE POINTS ON LISSAJOUS'S CURVES IN TWO AND THREE DIMENSIONS*

By Edward A. Hook

Lissajous's famous "Memoir sur l'étude optique des mouvements vibratoires," read before the Academie des Sciences in 1857, has aroused much interest in the study of harmonic curves — curves which result from the combination of two (or three) harmonic or vibratory motions at right angles to one another. Such a curve may be regarded as traced by a point whose rectangular coordinates, in two or three dimensions, are periodic functions of the time. If the periods of these functions are commensurable the curve is algebraic, if incommensurable it is non-algebraic. Only the first case will be considered here.

One of the most interesting problems connected with algebraic harmonic curves is the determination of the number of their multiple points. This problem was first studied in 1875 by Wilhelm Braun,† who determined the number and position of the double points on all such curves in two dimensions, and showed that double points are the only possible multiple points on the two-dimensional curve.

In 1897, Mr. E. H. Comstock‡ studied the same problem in both two and three dimensions, but confined his attention to the "open" curve, *i. e.* a curve which, for real values of the time parameter, breaks off abruptly at two "ends." Mr. Comstock's paper does not take into account the possible existence, in the case of three dimensions, of multiple points of multiplicity greater than two. His formula for the number of double points is correct only when the curve possesses no other multiple points than double points.

The purpose of the present paper is to develop the general formulae for the number and position of the finite real multiple points on any algebraic harmonic curve in both two and three dimensions. In the case of two dimen-

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† Inaugural-Dissertation der Philosophischen Facultät zu Erlangen. A summary of the same is given in *Math. Annalen*, Vol. 8, 1875.

‡ *Transactions Wisconsin Academy of Science*, 1897.

sions all of the results here given were obtained, though by a different method, by Braun. In the case of three dimensions the generality of the formulae obtained and the determination of the existence and number of quadruple points are believed to be new. Certain types of curves, for given period ratios, are finite in number. Where such is the case this number has been determined.

I. Plane Curves

1. Equations. The general equations of a harmonic curve of unit amplitudes generated by two harmonic motions at right angles to each other may be written:

$$x = \cos\left(\frac{\pi}{\lambda}s + C_1\right),$$

$$y = \cos\left(\frac{\pi}{\mu}s + C_2\right),$$

where s is the parameter and λ, μ, C_1, C_2 are constants. If we exclude the supposition that λ and μ are incommensurable, we can always, by a substitution of the form $t = \rho s$, reduce these equations to the form

$$x = \cos\frac{\pi}{\alpha}(t + t_1),$$

$$y = \cos\frac{\pi}{\beta}(t + t_2),$$

where α and β are integers prime to each other and $\pi t_1/\alpha = C_1$, $\pi t_2/\beta = C_2$.

Since, for real arguments, the cosine can vary only between $+1$ and -1, it is clear that for real values of t these equations can give only that part of the curve which lies within a square whose sides are parallel to the axes and at unit distance from the origin. Braun shows that the general or "closed" curve has no real branch outside of this square while the "open" curve has one real branch possessing a multiple point at infinity. We shall in all cases consider only the part of the curve within the above mentioned square.

In the work which follows we shall use t_0 to denote the value of t at any special point we are investigating and t to denote the general value of the parameter.

2. Construction. A convenient method for the approximate construction of these curves (when t_1 and t_2 are rational numbers) is the follow-

ing:—Suppose ϵ is the least common denominator of the fractions t_1 and t_2 so that $t_1 = t_1'/\epsilon$, $t_2 = t_2'/\epsilon$. Introducing the new variable $t' = \epsilon t$ we have

$$x = \cos \frac{\pi}{\epsilon a}(t' + t_1'),$$

$$y = \cos \frac{\pi}{\epsilon \beta}(t' + t_2').$$

Now, with the base of our square as a diameter, draw a semicircle and divide it into ϵa equal arcs. From each point of division construct perpendiculars to the X axis, thus dividing the square into ϵa rectangles. Similarly, draw a semicircle with one of the sides of the square as diameter and after dividing this into $\epsilon\beta$ equal arcs draw perpendiculars from these points of division to the Y axis. We have thus divided the square into $\epsilon^2 a\beta$ rectangles, the dis-

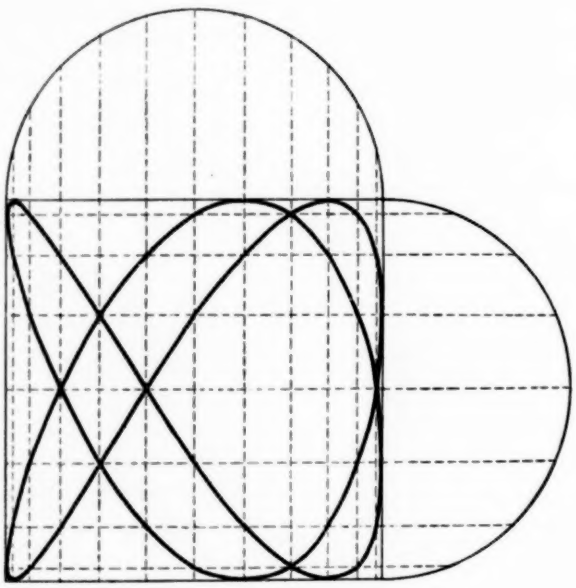

FIG. 1.

tances ot whose sides from the origin are cosines of integral values of $t' + t_1'$ and $t' + t_2'$. When $t' = 0$ the point (x, y) is a vertex of one of the rectangles. If t' be increased to unity the point becomes the opposite vertex of this

rectangle; when $t' = 2$ it becomes the opposite vertex of the first rectangle into which the tangent at the point for which $t' = 1$ extends; and so on. When t' has such a value that the vertex to which it corresponds lies on a side of the square, the curve, as we see by finding its slope, is tangent to the side of the square at this point. Further, it is convex toward the outside of the square. The expression for the slope, namely

$$\frac{dy}{dx} = \frac{\alpha}{\beta} \frac{\sin \frac{\pi}{\epsilon\beta}(t' + t'_2)}{\sin \frac{\pi}{\epsilon\alpha}(t' + t'_1)}$$

shows that the tangent can be parallel to a side of the square only when either $t' + t'_1 \equiv 0 \pmod{\epsilon\alpha}$ or $t' + t'_2 \equiv 0 \pmod{\epsilon\beta}$ and such values always make the cosine numerically unity, *i. e.* give us a point on the side of the square. Hence the method of procedure we have described will enable us to make an approximate construction for the whole curve by connecting the points so found by a continuous curve. Figure 1 shows the curve

$$x = \cos \frac{\pi}{3} t, \qquad y = \cos \frac{\pi}{2}(t + \tfrac{1}{4})$$

constructed in this way; here $\alpha = 3$, $\beta = 2$, $\epsilon = 4$.

The problem of finding the number of multiple points might, as was suggested to the writer by Professor H. S. White, be reduced to the following one, namely: Given a billiard table whose sides bear the ratio $\alpha : \beta$. A ball is placed at any point on the table and is shot in such a way that it strikes the side of the table at an angle of forty-five degrees; find the number of times it will cross its path before returning to its original position. The path will, of course, consist of straight lines each of which meets its successor at a right angle. The equations of this path are obtained from those of the curve by a transformation

$$u = \frac{\alpha}{\pi} \cos^{-1} x, \qquad (\alpha \geqq u \geqq 0)$$

$$v = \frac{\beta}{\pi} \cos^{-1} y, \qquad (\beta \geqq v \geqq 0)$$

u and v being the new coordinates. The broken-line path corresponding to the curve in Fig. 1 is shown in Fig 2.

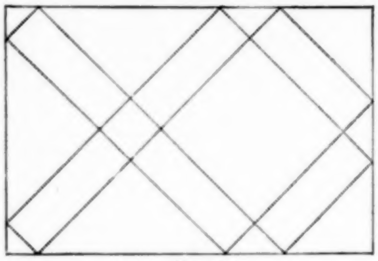

FIG. 2.

3. Periodicity. The necessary and sufficient condition that when θ is increased by the increment $\Delta\theta$ the function $\cos\theta$ shall be unchanged is that one of the two congruences

$$\Delta\theta \equiv 0 \pmod{2\pi}$$

or

$$\theta + \Delta\theta \equiv -\theta \pmod{2\pi}$$

be satisfied. Hence, if the abscissa x is to be unchanged when t is increased from t_0 to $t_0 + \Delta t$, we must have either

$$\frac{\pi}{a}\Delta t \equiv 0 \pmod{2\pi},$$

or

$$\frac{\pi}{a}\Delta t \equiv -2\frac{\pi}{a}(t_0 + t_1) \pmod{2\pi}.$$

Dividing through by π/a and also dividing the modulus by this we get

$$\Delta t \equiv 0 \pmod{2a},$$

or

$$\Delta t \equiv -2(t_0 + t_1) \pmod{2a}.$$

Similarly, the conditions that will cause y to be repeated are :—

$$\Delta t \equiv 0 \pmod{2\beta},$$

or

$$\Delta t \equiv -2(t_0 + t_2) \pmod{2\beta}.$$

Hence, when t is increased from t_0 to $t_0 + \Delta t$ there are just four conditions

under which the corresponding point on the curve will be brought back to its first position; these are:

$$
\begin{array}{llll}
(1) & \Delta t \equiv 0 \pmod{2\alpha}, & \Delta t \equiv 0 & \pmod{2\beta}; \\
(2) & \Delta t \equiv -2(t_0 + t_1) \pmod{2\alpha}, & \Delta t \equiv 0 & \pmod{2\beta}; \\
(3) & \Delta t \equiv 0 \pmod{2\alpha}, & \Delta t \equiv -2(t_0 + t_2) & \pmod{2\beta}; \\
(4) & \Delta t \equiv -2(t_0 + t_1) \pmod{2\alpha}, & \Delta t \equiv -2(t_0 + t_2) & \pmod{2\beta}.
\end{array}
$$

We see at once that it is always possible to satisfy (1) and that the smallest value of Δt which will do this is $\Delta t = 2\alpha\beta$. Now the direction cosines of the tangent at the point t_0 are proportional to

$$\left.\frac{dx}{dt}\right|_{t=t_0} = -\frac{\pi}{\alpha}\sin\frac{\pi}{\alpha}(t_0 + t_1) \quad \text{and} \quad \left.\frac{dy}{dt}\right|_{t=t_0} = -\frac{\pi}{\beta}\sin\frac{\pi}{\beta}(t + t_2),$$

and these are unchanged in value when we change t_0 to $t_0 + 2\alpha\beta$. We should find the same to be true of all of the higher derivatives of x and y with respect to t. Therefore, since the curve with which we are dealing is algebraic, we see that we are on the same branch of the curve on which we started and that if $\Delta t = 2\alpha\beta$ we have traced the whole curve at least once. We note also that we have proved that the curve cannot be tangent to itself at any point.

If (2) is to be satisfied we must have

$$\Delta t = -2(t_0 + t_1) + 2h\alpha = 2k\beta,$$

that is,

$$t_0 + t_1 = h\alpha - k\beta,$$

where h and k are integers. Since α and β are also integers the expression $h\alpha - k\beta$ will always be an integer. Hence, to satisfy (2) we must give t_0 certain special values.

The same is true of (3), as we learn by a similar process.

To satisfy (4) we must have

$$-2(t_0 + t_1) + 2h\alpha = -2(t_0 + t_2) + 2k\beta,$$

that is,

$$t_1 - t_2 = h\alpha - k\beta,$$

or,

$$t_1 \equiv t_2 \pmod{1},$$

a condition that is not in general satisfied. From what we now know of (1), (2), and (3) we know that if (4) is not satisfied, *the whole length of the curve is traversed just once when t receives the increment $2\alpha\beta$ and the curve is a closed curve.* For $2\alpha\beta$ is the least value of Δt that leaves both point and

slope unchanged. If (4) can be satisfied this is not the case. Unlike (2) and (3), if (4) can be satisfied at all it can be satisfied for every point of the curve. The smallest value of Δt that will do this is in general less than $2\alpha\beta$. When t is given such an increment the direction cosines are unchanged in absolute value but both are changed in sign. This leads to the conclusion that we are simply retracing the curve in the opposite direction. Since now $t_1 \equiv t_2 \pmod 1$, it is possible to find two values of t_0 such that $t_0 + t_1 \equiv 0 \pmod \alpha$ and $t_0 + t_2 \equiv 0 \pmod \beta$ simultaneously. But such a point must lie at a vertex of the square within which the part of the curve we are considering is confined. Conversely we may prove that if the curve has a point at a vertex of the square, the condition $t_1 \equiv t_2 \pmod 1$ is satisfied. Now both x and y are numerically unity at such a point. If we give t a small (real) increment Δt, the values of x and y obtained will be the same whether Δt be positive or negative. Hence we begin to retrace the curve at the vertex of the square and such a point may be called an "end" of the curve. *Such a curve is the "open" form of the Lissajous's curve.*

4. Multiple Points. *General.* If t_0 is a point which makes (2) possible, and the smallest value of Δt that will satisfy (2) is found, the addition of this Δt to the parameter will not affect the value of dy/dt. But usually dx/dt will have been changed in sign though not in numerical value. Hence the direction of the tangent will have been changed and since the curve is algebraic, it must cross itself at this point. Hence values of t_0 which make (2) solvable belong in general to multiple points. The same is true of (3). Since the absolute values of the direction cosines are always the same at such a point there are only two different directions which the tangent at the point can assume according as the two direction cosines have like or unlike signs. Whence it follows that *a multiple point of order higher than the second is impossible*, and that the slope of one branch of the curve at a double point is always the negative of the slope of the other branch.

Closed Curves. We have seen in §3, that if (2) is satisfied we have

$$t_0 + t_1 = h\alpha - k\beta.$$

The second member is always an integer and may be any integer whatever, since α and β have no common factor. Hence the necessary and sufficient condition that (2) be satisfied is that $t_0 + t_1$ be an integer. Between zero and $2\alpha\beta$ there are $2\alpha\beta$ values of t_0 which satisfy (2). But not all of these determine double points. If $t_0 + t_1 \equiv 0 \pmod \alpha$, condition (2) becomes iden-

tical with (1). The curve then has but one tangent at the point in question and since we have shown in §3 that the curve cannot be tangent to itself such a point cannot be a double point. This occurs 2β times. All of the other values must give double points. Now we have counted a value of t_0 for each branch of the curve through the double point. Hence (2) determines $\frac{1}{2}(2\alpha\beta - 2\beta) = \beta(\alpha - 1)$ double points.

If we seek the number of values of $t_0 + t_1$ less than $2\alpha\beta$ and such that $t_0 + t_1 \equiv n \pmod{\alpha}$, where n is any one of the integers $1, 2, 3, \ldots (\alpha - 1)$, we find it to be just 2β, no matter what the value of n. Hence the double points determined by (2) lie in groups of β on $\alpha - 1$ lines parallel to the y axis and corresponding to integral values of $t_0 + t_1$.

Similarly we find that (3) determines $\alpha(\beta - 1)$ double points lying in groups of α on $\beta - 1$ lines parallel to the x axis and corresponding to integral values of $t_0 + t_2$.

Since $t_1 - t_2$ is not an integer, (2) and (3) cannot be solved simultaneously and hence the double points thus far determined are the only ones the curve can possess. If we call lines parallel to either axis and corresponding to integral values of $t + t_1$, other than zero or α, and of $t + t_2$ other than zero or β, integer lines, we have the following

THEOREM. *Closed curves possess $\beta(\alpha - 1) + \alpha(\beta - 1)$ real double points which fall into two classes: (1°) $\beta(\alpha - 1)$ lying in groups of β on the $\alpha - 1$ integer lines parallel to the y axis, and, (2°), $\alpha(\beta - 1)$ lying in groups of α on the $\beta - 1$ integer lines parallel to the x axis.*

Open Curves. Here $t_1 \equiv t_2 \pmod{1}$. Since now we trace out the whole curve twice when we let Δt increase from zero to $2\alpha\beta$, we must divide the number of values of t_0 corresponding to double points by four instead of by two. The condition that (2) be solvable is that $t_0 + t_1$ be an integer. Likewise the condition that (3) be solvable is that $t_0 + t_2$ be an integer. But if either of these expressions is an integer the other must be also for their difference, $t_1 - t_2$, is an integer. Hence (2) and (3) are simultaneously satisfied. This occurs $2\alpha\beta$ times. But in 2β cases (2) reduces to (1) and in 2α cases (3) does so. The two do so together twice. When either of these cases occurs one of the direction cosines is zero and hence a change in the sign of the other does not affect the direction of the tangent. In such a case, therefore, we fail to get a double point. Hence:

THEOREM. *Open curves possess $\frac{1}{2}(\alpha - 1)(\beta - 1)$ double points and these lie at the alternate intersections of the $\alpha - 1$ integer lines parallel to the y axis with the $\beta - 1$ integer lines parallel to the x axis.*

That the use of the word "alternate" in this theorem is correct will appear at once if we construct the open curve by the method of § 2. For suppose at a given double point $t_0 + t_1$ and $t_0 + t_2$ are two integers a and b. Then $a - b = t_1 - t_2$, a constant for every point on the curve. But to make the curve pass through an adjacent point of intersection of one of the integer lines through the given point with an integer line parallel to the other we must have, say, $t_0 + t_1 = a + 1$ and $t_0 + t_2 = b$. That is, one of these values must be changed by unity while the other remains unchanged. But at such a point we should have $t_1 - t_2 = a - b + 1$ and this cannot occur for any point on the given curve.

5. Number of Curves. The question as to whether or not the number of curves of the two types is finite or infinite presents some points of interest. In the case of closed curves the only restriction on the values of t_1 and t_2 is that their difference must not be an integer. We have no right to conclude, however, that any two pairs of values of t_1 and t_2 whose differences are neither integers nor equal will give the equations of different curves. It might be that the two pairs of equations would yield the same sets of points for different values of the parameter. Suppose two such sets of values are t_1', t_2', and t_1'', t_2''. Let us choose two values t_0' and t_0'' such that $t_0' = -t_1'$ and $t_0'' = -t_1''$. This makes the two points under consideration points on the side $x = 1$ of the square and removes all possibility of complications with double points or points of intersection of the two curves. The two values of y are $y' = \cos[\pi(t_0' + t_2')/\beta]$ and $y'' = \cos[\pi(t_0'' + t_2'')/\beta]$. If the two points lie on the same curve it must be possible to find a value of Δt that will leave x unchanged and change y' into y''. For this we must have

$$\Delta t \equiv 0 \qquad (\text{mod } 2\alpha),$$

$$\Delta t \equiv (t_0'' + t_2'') - (t_0' + t_2') \ (\text{mod } 2\beta);$$

that is,

$$\Delta t = 2h\alpha = 2k\beta + (t_0'' + t_2'') - (t_0' + t_2').$$

This can occur only when $(t_0'' + t_2'') - (t_0' + t_2')$ is an even integer. Hence if this difference is not an even integer the curves are distinct. Since there are an infinite number of sets of values for which this difference is not such an integer there are an infinite number of closed curves.

The number of open curves we determine at once from the fact that each open curve passes through two vertices of the square. Since no two curves for which α and β are the same can pass through the same vertex we see that there are just two open curves.

II. Space Curves

6. Equations. The general equations of a harmonic curve of unit amplitudes generated by three harmonic motions at right angles to each other may be written:

$$x = \cos\left(\frac{\pi}{\lambda}s + C_1\right),$$

$$y = \cos\left(\frac{\pi}{\mu}s + C_2\right),$$

$$z = \cos\left(\frac{\pi}{\nu}s + C_3\right),$$

where s is the parameter and λ, μ, ν, C_1, C_2, C_3 are constants. We exclude the cases in which any one of the ratios λ/μ, μ/ν, ν/λ is irrational. If λ, μ, ν are not integers, we may make them such by the substitution $t = \rho s$. If ρ is suitably chosen, our new λ, μ, ν are now integers possessing no common factor. Let a be the highest common factor of μ and ν, b that of ν and λ, and c that of λ and μ. Then our equations may be put in the form:

$$(X) \qquad x = \cos\frac{\pi}{bc\alpha}\left(t + t_1\right),$$

$$(Y) \qquad y = \cos\frac{\pi}{ca\beta}\left(t + t_2\right),$$

$$(Z) \qquad z = \cos\frac{\pi}{ab\gamma}\left(t + t_3\right),$$

where $\dfrac{\pi t_1}{bc\alpha} = C_1$, $\dfrac{\pi t_2}{ca\beta} = C_2$, $\dfrac{\pi t_3}{ab\gamma} = C_3$. We note that no two of the quantities a, b, and c have a common factor and that the same is true of α, β, and γ. Moreover α is prime to a, β to b, and γ to c. But t_1, t_2, t_3 may be integral, fractional, or irrational.

The part of the curve given by real values of t is confined to the interior of a cube whose faces lie parallel to the coordinate planes and at unit distance from the origin. Our results will apply only to this part of the curve. It would not be difficult to show that any singularities which the curve may possess outside of this cube must be imaginary if the curve be "closed" and must lie at infinity if the curve be "open."

We note for future reference that, for a given value of t, each of the equations (X), (Y), (Z) represents a plane perpendicular to the X, Y, Z axis respectively.

7. Periodicity. The same reasoning as that used with plane curves shows that there are the following eight possible ways in which we may be brought back to a given point, t_0, by increasing t to $t_0 + \Delta t$:

$$
\begin{array}{ll}
(1) & \Delta t \equiv 0 \\
(2) & \Delta t \equiv -2(t + t_1) \\
(3) & \Delta t \equiv 0 \\
(4) & \Delta t \equiv 0 \\
(5) & \Delta t \equiv 0 \\
(6) & \Delta t \equiv -2(t + t_1) \\
(7) & \Delta t \equiv -2(t_0 + t_1) \\
(8) & \Delta t \equiv -2(t_0 + t_1)
\end{array}
\left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} (\mathrm{mod}\ 2\,bca)
\quad
\begin{array}{l}
\Delta t \equiv 0 \\
\Delta t \equiv 0 \\
\Delta t \equiv -2(t_0 + t_2) \\
\Delta t \equiv 0 \\
\Delta t \equiv -2(t_0 + t_2) \\
\Delta t \equiv 0 \\
\Delta t \equiv -2(t_0 + t_2) \\
\Delta t \equiv -2(t_0 + t_2)
\end{array}
\left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} (\mathrm{mod}\ 2\,ca\beta)
\quad
\begin{array}{l}
\Delta t \equiv 0 \\
\Delta t \equiv 0 \\
\Delta t \equiv 0 \\
\Delta t \equiv -2(t_0 + t_3) \\
\Delta t \equiv -2(t_0 + t_3) \\
\Delta t \equiv -2(t_0 + t_3) \\
\Delta t \equiv 0 \\
\Delta t \equiv -2(t_0 + t_3)
\end{array}
\left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} (\mathrm{mod}\ 2\,ab\gamma)
$$

We notice first of all that (1) is the only one of these systems of congruences that is solvable for every curve and every point on the curve. It is furthermore the only one that leaves the direction cosines, which are proportional to

$$\frac{dx}{dt} = -\frac{\pi}{bca} \sin \frac{\pi}{bca}(t + t_1),$$

$$\frac{dy}{dt} = -\frac{\pi}{ca\beta} \sin \frac{\pi}{ca\beta}(t + t_2),$$

$$\frac{dz}{dt} = -\frac{\pi}{ab\gamma} \sin \frac{\pi}{ab\gamma}(t + t_3),$$

unchanged. We should find, on investigating, that all higher derivatives of x, y, and z with respect to t are likewise unchanged. This proves, first, that we have been brought back along the branch of the curve on which we started and, second, that the curve cannot be tangent to itself at any point. Now the smallest value of Δt that will satisfy (1) is $2abca\beta\gamma$. Hence the curve is traced completely at least once when t receives an increment $2abca\beta\gamma$.

To satisfy any one of the conditions (2) – (7) we must impose special values on t_0, as we see at once if we write them in equation form.

To satisfy (8) we must have

$$\Delta t = -2(t_0 + t_1) + 2hbca = -2(t_0 + t_2) + 2kca\beta = -2(t_0 + t_3) + 2lab\gamma$$

and if this is satisfied we must have

$$t_1 \equiv t_2 \pmod{c}, \qquad t_2 \equiv t_3 \pmod{a}, \qquad t_3 \equiv t_1 \pmod{b};$$

and, conversely, if these congruences are satisfied, (8) can always be satisfied *whatever the value of* t_0. If (8) cannot be satisfied, giving t the increment $2abca\beta\gamma$ must cause the whole length of the curve to be traced out just once since this is the least value that will satisfy (1) and since no other of the conditions (1)–(7) can be satisfied except for special points. But if (8) can be satisfied, then there is, in general, a value of Δt less than $2abca\beta\gamma$ which will bring us back to our starting point and leave the tangent at the point unchanged, and this value is the least value of Δt that will satisfy (8). For if Δt have such a value the direction cosines are all changed in sign but are unchanged in absolute value.

We easily find that the condition

$$t_0 + t_1 \equiv 0 \pmod{bca}, \quad t_0 + t_2 \equiv 0 \pmod{ca\beta}, \quad t_0 + t_3 \equiv 0 \pmod{ab\gamma},$$

can be satisfied when and only when (8) is solvable, and in this case is satisfied for two points on the curve. These points are vertices of the cube. Considerations entirely similar to those used in the corresponding case in plane curves show that we are now dealing with "open curves" in three dimensions. All other curves must be "closed" curves. The condition $t_1 \equiv t_2 \pmod{c}$ is easily found to be necessary and sufficient to make the projection of the curve on the XY plane an open curve. The methods used in investigating plane curves will show this. Similarly $t_2 \equiv t_3 \pmod{a}$ and $t_3 \equiv t_1 \pmod{b}$ are the conditions that the YZ and ZX projections, respectively, be open curves.

8. Classification. These facts afford us a convenient means of classifying our curves. We may distinguish four types, as follows :—

A. The general curve. Projections all closed curves.
B. Curves possessing one open-curve projection.
C. Curves possessing two open-curve projections.
D. Curves possessing three open-curve projections.

9. Multiple Points in General. If any one of the sets of congruences (2) to (7) inclusive can be satisfied, and we give Δt a value that does this, having first found a point t_0 that makes it possible to find such a Δt, then the direction cosines at the point in question are unchanged in absolute value. But usually one or two of them will be changed in sign. When this happens

the second value of t gives a tangent to the curve different from that given by the first. The curve being algebraic, such a point must be a multiple point.

Conditions (2), (3), (4) turn out to be solvable for points on all four types of curves. Hence a consideration of these should give us something of general value in determining the number of multiple points.

From the last two congruences in (2) we get $\Delta t = 2kabc\beta\gamma$ and from the first $\Delta t = -2(t_0 + t_1) + 2hbc\alpha$. Hence, if (2) is satisfied, we must have $kabc\beta\gamma = -(t_0 + t_1) + hbc\alpha$; whence $t_0 + t_1 = hbc\alpha - kabc\beta\gamma$. The second member of this equation will always be a multiple of bc, and, by a proper choice of h and k, may be made to equal any such multiple. Hence, if (2) is satisfied, we must have $t_0 + t_1 \equiv 0 \pmod{bc}$. This is satisfied $2a\alpha\beta\gamma$ times between the values $t_0 = 0$ and $t_0 = 2abc\alpha\beta\gamma$. But in some of these cases (2) becomes identical with (1). This occurs $2\alpha\beta\gamma$ times, viz. when $t_0 + t_1 \equiv 0 \pmod{bc\alpha}$. These we must subtract. Now in counting these $2\alpha\beta\gamma(a-1)$ values of t_0 we have counted one for each branch of the curve through a given point. Through any such point there are two and only two branches *defined by condition* (2). For if we give t the increment indicated by (2) twice, we restore the original state of affairs exactly, as far as geometric conditions go. Hence (2) determines $\alpha\beta\gamma(a-1)$ values of t_0 of which we must take account in enumerating multiple points.

Similarly (3) and (4) determine $b\gamma\alpha(\beta - 1)$ and $c\alpha\beta(\gamma - 1)$ such points respectively.

If (5) is to be satisfied, we must have

$$2hbc\alpha = -2(t_0 + t_2) + 2kca\beta = -2(t_0 + t_3) + 2lab\gamma.$$

The necessary and sufficient condition for this is that $t_2 \equiv t_3 \pmod{a}$; for if this be the case we can determine t_0, k, and l so that the second and third members of the equation shall be equal multiples of $2bc\alpha$ and thus determine h; and conversely if $t_2 - t_3$ is not a multiple of a we can never make the second and third members equal and hence (5) cannot be satisfied. Now, as we have seen, $t_2 \equiv t_3 \pmod{a}$ is simply the condition that the YZ projection of the curve be an open curve. Therefore if (5) is satisfied, the YZ projection will be an open curve. Similarly (6) can be satisfied when, and only when, the ZX projection is an open curve, *i. e.* when $t_3 \equiv t_1 \pmod{b}$; and (7) can be satisfied when, and only when, the XY projection is an open curve, *i. e.* when $t_1 \equiv t_2 \pmod{c}$.

We shall now take up the four types separately.

10. Multiple Points on Curves of Type A. For these curves none of the conditions (5), (6), (7), (8) can be satisfied. But it is conceivable that two or more of the conditions (2), (3), (4) might be simultaneously satisfied, thus indicating the existence of multiple points of an order higher than the second. But if (2) and (3) could be satisfied at the same point, we should have $t_1 \equiv t_2 (\mathrm{mod}\ c)$, which is not the case. Similarly, neither (2) and (4) nor (3) and (4) can be simultaneously satisfied. Hence the general curve possesses only multiple points of the second order and their number is $a\beta\gamma(a-1) + b\gamma a(\beta - 1) + ca\beta(\gamma - 1)$.

We have seen that if (2) is satisfied we have $t_0 + t_1 \equiv 0\ (\mathrm{mod}\ bc)$. But if $t_0 + t_1 \equiv 0\ (\mathrm{mod}\ bca)$ we do not get multiple points. The only points corresponding to the latter values lie on the two faces, $x = \pm 1$, of the cube. If $t_0 + t_1 \equiv nbc (\mathrm{mod}\ bca)$, where n is an integer such that $0 < n < a$, there are just $2a\beta\gamma$ values of t_0 less than $2abca\beta\gamma$ that will satisfy this and hence just $a\beta\gamma$ double points lie in each plane (X) corresponding to integral values of $(t + t_1)/bc$. Like statements can be proved about the double points determined by (3) and (4). Hence:—

THEOREM. *The double points on the general curve lie on three sets of planes; namely, $a\beta\gamma(a-1)$ of them in groups of $a\beta\gamma$ on $a-1$ planes (X) corresponding to integral values of $(t + t_1)/bc$, $ba\gamma(\beta-1)$ in groups of $ba\gamma$ on $\beta - 1$ planes (Y) corresponding to integral values of $(t + t_2)/ca$, and $ca\beta(\gamma - 1)$ in groups of $ca\beta$ on $\gamma - 1$ planes (Z) corresponding to integral values of $(t + t_3)/ab$, and no two of these can coincide.*

11. Multiple Points on Curves of Type B. Suppose $t_1 \equiv t_2$ (mod c). (2) and (3) can now be satisfied simultaneously. We know, (§9), that for any point satisfying (2) we have $t_0 + t_1 \equiv 0 (\mathrm{mod}\ bc)$ and for any point satisfying (3) we have $t_0 + t_2 \equiv 0\ (\mathrm{mod}\ ca)$. There are $2a\beta\gamma$ values of t_0 less than $2abca\beta\gamma$ which satisfy both of these conditions. A glance at (2) and (3) shows that there must in general be four separate sets of direction cosines at one of these points. For neither the original nor the final sets can be the same and giving t the indicated increment changes both of the original sets. Hence, in general, these points are quadruple points. But if it happen that either $t_0 + t_1 \equiv 0\ (\mathrm{mod}\ bca)$ or $t_0 + t_2 \equiv 0\ (\mathrm{mod}\ ca\beta)$, then (2) in the first case, and (3) in the second, reduces to (1) and we have merely a double point, for then only two directions for the tangent line are possible. If (2) and (3) reduce to (1) at the same point we have not a multiple point at all. Among our $2a\beta\gamma$ values of t_0, $2\beta\gamma$ make (2) reduce to (1) and $2\gamma a$ make (3) do so, while

2γ of these make (2) and (3) do so together. Hence we have a total of $\frac{1}{4}(2\alpha\beta\gamma - 2\beta\gamma - 2\gamma\alpha + 2\gamma) = \frac{1}{2}(\alpha - 1)(\beta - 1)\gamma$ quadruple points. (We divide by 4 because there is a value of t_0 for each branch through a quadruple point.)

Now each value of t_0 yielding a quadruple point has been counted twice among our $2(a + b + c)\alpha\beta\gamma - 2a\beta\gamma - 2b\gamma\alpha - 2c\alpha\beta$ values which (2), (3), (4) together give. It has been counted once under (2) and once under (3). Hence (2), (3), (4) determine

$$(a + b + c - 2)\alpha\beta\gamma - (a - 2)\beta\gamma - (b - 2)\gamma\alpha - c\alpha\beta - 2\gamma$$

double points.

So far we have left out of account conditions (5) – (8). We have seen, (§9), that (7) can be satisfied and that (5), (6), (8) cannot. If (7) is satisfied we have $t_0 + t_1 \equiv 0 \pmod{b}$ and $t_0 + t_2 \equiv 0 \pmod{a}$. This occurs $2c\alpha\beta\gamma$ times. But if $t_0 + t_1 \equiv 0 \pmod{bc}$, (7) reduces to (2), and if $t_0 + t_2 \equiv 0 \pmod{ca}$, (7) reduces to (3). There are $2\alpha\beta\gamma$ cases of either of these. However, if $t_0 + t_1 \equiv 0 \pmod{bc}$ when $t_0 + t_2 \equiv 0 \pmod{a}$, it follows, since $t_1 \equiv t_2 \pmod{c}$, that $t_0 + t_2 \equiv 0 \pmod{ca}$. Hence, these exceptional cases under (7) having already been taken account of elsewhere, must be subtracted from our $2c\alpha\beta\gamma$ values, but need only be subtracted once. Consequently, (7) determines $c\alpha\beta\gamma - \alpha\beta\gamma$ additional double points. Curves of this type therefore possess $(a + b + 2c - 3)\alpha\beta\gamma - (a - 2)\beta\gamma - (b-2)\gamma\alpha - c\alpha\beta - 2\gamma$ double points.

From the congruence conditions for this type of curves we read off at once that all quadruple points must lie on the lines of intersection of two sets of planes (X) and (Y) corresponding to integral values of $(t + t_1)/bc$ and $(t + t_2)/ca$ respectively and that they reduce to double points if the lines in question lie on a face of the cube and to simple points if these lines are edges of the cube. If we fix a pair of integers, say m, n, and seek the number of values of t_0 less than $2abc\alpha\beta\gamma$ that make $t_0 + t_1 \equiv m \pmod{bc\alpha}$ and $t_0 + t_2 \equiv n \pmod{c\alpha\beta}$, we get 2γ as the result. Since there are just two possible open-curve projections on the XY plane and the cylinders whose sections these are have double lines in alternate lines of the $(\alpha - 1)(\beta - 1)$ possible double lines the quadruple points must all lie on one set of $\frac{1}{2}(\alpha - 1)(\beta - 1)$ of these. Hence on each line there lie γ quadruple points.

THEOREM. *The* $\frac{1}{2}\gamma(\alpha - 1)(\beta - 1)$ *quadruple points on curves whose*

XY projection is an open curve lie in groups of γ *on* $\frac{1}{2}(\alpha - 1)(\beta - 1)$ *alternate lines of intersection of planes* (X) *and* (Y) *corresponding to integral values of* $(t_0 + t_1)/bc$ *and* $(t_0 + t_2)/ca$ *respectively.*

The remaining multiple points determined by (2), (3), (4) are all double points and their distribution is similar to that of the double points in curves of type A. We have only to remember that the quadruple points now take the place of some of the double points in type A. The present type differs from A, however, in that some of the points may lie upon faces of the cube. The latter are those points not on an edge of the cube which would be quadruple points if they lay within the cube.

In addition we have the new double points determined by (7). Suppose m and n are an arbitrary pair of integers. Then, as we have found before, there are, if any, 2γ values of t_0 less than $2abc\alpha\beta\gamma$ that will give $t_0 + t_1 \equiv m$ $(\text{mod } bc\alpha)$ and $t_0 + t_2 \equiv n(\text{mod } ca\beta)$, *i. e.*, the curve intersects such a line, if at all, 2γ times. This is true if the line is one of those which pass through points satisfying (7), though now the points fall together in pairs. Hence double points determined by (7) lie in groups of γ on $(c - 1)\alpha\beta$ lines of intersection of planes (X) and (Y) for which $(t + t_1)/b$ and $(t + t_2)/a$ respectively are integers. But how are these lines distributed? The XY projection is divided by the planes for which $t_0 + t_1 \equiv 0(\text{mod } bc)$ and $t_0 + t_2 \equiv 0$ $(\text{mod } ca)$ into $\alpha\beta$ rectangles. Let us seek the number of the above lines whose projections lie in one of these rectangles. At such a point we have $t_0 + t_1 \equiv r$ (mod bc) and $t_0 + t_2 \equiv s(\text{mod } ca)$, r and s being values of $t_0 + t_1$ and $t_0 + t_2$ which give one of the points in question. Apparently there are c values of r, and c values of s that satisfy the conditions. This would make c^2 pairs of values of r and s. But $t_1 \equiv t_2$ (mod c) and hence necessarily $r \equiv s$ (mod c). This reduces the number of pairs to c. One of these pairs makes $t_0 + t_1 \equiv 0$ (mod bc) and $t_0 + t_2 \equiv 0$ (mod ca). Hence, within each of the $\alpha\beta$ rectangles there are $c - 1$ points which are projections of the lines in question.

THEOREM. *The* $(c - 1)\alpha\beta\gamma$ *additional double points indicated by* (7) *lie in groups of* γ *on* $(c - 1)\alpha\beta$ *lines of intersection of planes* (X) *and* (Y) *for which* $(t + t_1)/b$ *and* $(t + t_2)/a$ *respectively are integers and these lines lie in groups of* $c - 1$ *within and parallel to the sides of the parallelopipeds whose ends lie in the planes* $z = \pm 1$ *and whose sides are the planes* (X) *and* (Y) *for which* $(t + t_1)/bc$ *and* $(t + t_2)/ca$, *respectively, are integers.*

A similar set of theorems will of course hold for curves of type B in case we have in place of $t_1 \equiv t_2 \pmod{c}$, either $t_3 \equiv t_1 \pmod{b}$ or $t_2 \equiv t_3 \pmod{a}$.

12. Multiple Points on Curves of Type C. Suppose $t_1 \equiv t_2 \pmod{c}$ and $t_3 \equiv t_1 \pmod{b}$. We can now satisfy simultaneously (2) and (3) and also (2) and (4), but not (3) and (4). By reasoning similar to that used in the consideration of curves of type B, we learn that there are $\frac{1}{2}(\alpha - 1)(\beta - 1)\gamma$ quadruple points at which (2) and (3) are identical and $\frac{1}{2}(\gamma - 1(\alpha - 1)\beta$ at which (2) and (4) are identical.

This leaves

$$(a + b + c - 4)\alpha\beta\gamma - (a - 4)\beta\gamma - (b - 2)\gamma\alpha - (c - 2)\alpha\beta - 2\beta - 2\gamma$$

double points determined by (2)(3), and (4).

We can now satisfy both (6) and (7) but never simultaneously, for $t_2 \equiv t_3 \pmod{a}$ is the condition for this and this congruence does not hold. By reasoning like that used for type B, we find that (6) and (7) determine $(b + c - 2)\alpha\beta\gamma$ additional double points. Hence curves of this type possess $\frac{1}{2}[(\alpha - 1)(\beta - 1)\gamma + (\gamma - 1)(\alpha - 1)\beta]$ quadruple points and

$$(a + 2b + 2c - 6)\alpha\beta\gamma - (a - 4)\beta\gamma - (b - 2)\gamma\alpha - (c - 2)\alpha\beta - 2\beta - 2\gamma$$

double points.

The theorems regarding the position and distribution for this type of curve are seen at once to be very similar to those for type B. We have only to consider two sets of points in each case instead of one and to make the proper changes in the letters of our formulae.

If instead of the conditions $t_1 \equiv t_2 \pmod{c}$ and $t_3 \equiv t_1 \pmod{b}$ we have $t_1 \equiv t_2 \pmod{c}$ and $t_2 \equiv t_3 \pmod{a}$ or $t_3 \equiv t_1 \pmod{b}$ and $t_2 \equiv t_3 \pmod{a}$, the corresponding results are obtained by simply advancing the letters in cyclic rotation.

13. Multiple Points on Curves of Type D. Here $t_1 \equiv t_2 \pmod{c}$, $t_2 \equiv t_3 \pmod{a}$, $t_3 \equiv t_1 \pmod{b}$. We have seen that (8) is now satisfied by a value of Δt less than $2abc\alpha\beta\gamma$. Hence when $\Delta t = 2abc\alpha\beta\gamma$ *we have traversed the whole length of the curve twice.* Hence, in counting multiple points, we must divide the number of values of t_0 less than $2abc\alpha\beta\gamma$ which give quadruple points by eight, and the number which give double points by four.

As before there are $2(a + b + c)\alpha\beta\gamma - 2a\beta\gamma - 2b\gamma\alpha - 2c\alpha\beta$ values of t_0 satisfying (2), (3), (4). But we must be cautious how we treat these until we find whether or not the same values are given again in some other way.

However, by a comparison of (2) and (5) we find that the conditions that these be satisfied simultaneously are fulfilled and that this is true for every point which satisfies either of them. The same is true of (3) and (6) and of (4) and (7). Hence to find the number of quadruple points on the curve we have only to find the number of values of t_0 which make two of the conditions (2), (3), and (4) identical, and then divide by eight. If (2) and (3) are both satisfied at once we have $t_0 + t_1 \equiv 0 \pmod{bc}$ and $t_0 + t_2 \equiv 0 \pmod{ca}$. But since $t_1 \equiv t_3 \pmod{b}$ and $t_2 \equiv t_3 \pmod{a}$ we have also $t_0 + t_3 \equiv 0 \pmod{ab}$, the condition that (4) be satisfied. Hence if any two of (2), (3), (4) are satisfied the third is also. Now (2), (3), (4) are satisfied together $2\alpha\beta\gamma$ times. But what if (2), say, reduces to (1)? This means that $t_0 - t_1 \equiv 0$ (mod $2bca$) and hence the x direction cosine in (2) is zero and is unchanged when Δt increases so as to bring us back to the starting point. Since (3) and (4) hold simultaneously with (2), the same is true of them. But this makes possible only two distinct sets of direction cosines at the point in question and hence this is not a quadruple point but merely a double point. Similarly we may show that if two of (2), (3), (4) reduce to (1) at the same time we have merely an ordinary point of the curve. Now (2) reduces to (1), $2\beta\gamma$ times, (3) does so $2\gamma\alpha$ times and (4) $2\alpha\beta$ times among our $2\alpha\beta\gamma$ values of t_0; (2) and (3) do so together 2γ times, (2) and (4), 2β times, and (3) and (4), 2α times; (2), (3), and (4) together reduce to (1) twice among these. Hence the number of values of t_0 which really indicate quadruple points is $2\alpha\beta\gamma - 2\beta\gamma - 2\gamma\alpha - 2\alpha\beta + 2\alpha + 2\beta + 2\gamma - 2$. Dividing by eight and factoring we have

$$\tfrac{1}{4}(\alpha - 1)(\beta - 1)(\gamma - 1) \text{ quadruple points.}$$

Now in general we have $2(a + b + c)\alpha\beta\gamma - 2a\beta\gamma - 2b\gamma\alpha - 2c\alpha\beta$ values of t_0 which satisfy (2), (3), or (4), when these do not reduce to (1). From these we must subtract three times the number which go to determine quadruple points. For we have counted each such value as if it belonged separately to (2), (3), and (4). But in our general number of values of t_0 we have also counted twice those values which make either (2), (3), or (4) separately become (1). These should be counted only once. Bearing this in mind we can now write down the number of double points on the curve. It is:—

$$\tfrac{1}{2}[(a+b+c-3)\alpha\beta\gamma-(c-2)\alpha\beta-(b-2)\gamma\alpha-(a-2)\beta\gamma-2\alpha-2\beta-2\gamma+3].$$

The conditions for a quadruple point on the open curve show at once

that every quadruple point lies at the intersection of three planes $(X), (Y)$, and (Z) such that $(t + t_1)/bc$, $(t + t_2)/ca$, and $(t + t_3)/ab$ are all integers. But no quadruple point can lie on a face of the cube. By a method entirely similar to that used in showing that the double points on the open plane curve must lie at alternate intersections of pairs of integer lines we can show here that no two such points that are adjacent,—*i. e.* lie on the same edge of one of the parallelopipeds into which the planes of the above three kinds divide the cube,—can lie on the same curve. There are just $(\alpha - 1)(\beta - 1)(\gamma - 1)$ vertices of such parallelopipeds within the cube. The theorem just stated shows that no one curve can have quadruple points at more than two vertices of one of these parallelopipeds and that these two vertices must be opposite,—*i. e.* not on the same edge. But there are four such pairs of vertices for each parallelopiped. Hence we might expect four open curves and we shall see later that this is the case. If we use the word "alternate" in a somewhat loose sense to describe such a distribution of points we have the

THEOREM. *The* $\frac{1}{4}(\alpha - 1)(\beta - 1)(\gamma - 1)$ *quadruple points lie at alternate points of intersection of three sets of planes* $(X), (Y)$, *and* (Z) *such that* $(t + t_1)/bc$, $(t + t_2)/ca$, *and* $(t + t_3)/ab$ *are all integers.*

The double points on this type of curve fall into two classes. The first class includes those points on faces of the cube, but not on edges, which would be quadruple points if they lay within the cube. There is no difficulty in proving the

THEOREM. $\frac{1}{4}(\alpha - 1)(\beta - 1)$ *double points lie at intersections of each of the planes* $z = \pm 1$ *with the set of interior planes* (X) *and* (Y) *for which* $(t + t_1)/bc$, $(t + t_2)/ca$ *are integers*; $\frac{1}{4}(\gamma - 1)(\alpha - 1)$ *lie at intersections of each of the planes* $y = \pm 1$ *with the set of interior planes* (X) *and* (Z) *for which* $(t + t_1)/bc$, $(t + t_3)/ab$ *are integers*; *and* $\frac{1}{4}(\beta - 1)(\gamma - 1)$ *lie at intersections of each of the planes* $x = \pm 1$ *with the set of interior planes* (Y) *and* (Z) *for which* $(t + t_2)/ca$, $(t + t_3)/ab$ *are integers.*

The second class of double points includes those which lie within the cube. If we remember that they occur when either (2) and (5), (3) and (6), or (4) and (7), are satisfied together we are able to prove by familiar methods the following

THEOREM: *The double points lying within the cube consist of* (1°) $\frac{1}{2}(c - 1)\alpha\beta(\gamma - 1)$ *points which are one half of the* $(c - 1)\alpha\beta(\gamma - 1)$ *points that lie in groups of* $\gamma - 1$ *on* $(c - 1)\alpha\beta$ *lines of intersection of planes* (X) *and* (Y) *for which* $(t + t_1)/b$ *and* $(t + t_2)/a$ *are integers, these lines lying in*

SUMMARY FOR SPACE CURVES

OPEN - CURVE PROJECTIONS	NUMBER OF QUADRUPLE POINTS	NUMBER OF DOUBLE POINTS
None	*None*	$(a+b+c)\alpha\beta\gamma-a\beta\gamma-b\gamma\alpha-c\alpha\beta$
XY	$\frac{1}{2}(\alpha-1)(\beta-1)\gamma$	$(a+b+2c-3)\alpha\beta\gamma-(a-2)\beta\gamma-(b-2)\gamma\alpha-c\alpha\beta-2\gamma$
YZ	$\frac{1}{2}(\beta-1)(\gamma-1)\alpha$	$(b+c+2a-3)\alpha\beta\gamma-(b-2)\gamma\alpha-(c-2)\alpha\beta-a\beta\gamma-2\alpha$
ZX	$\frac{1}{2}(\gamma-1)(\alpha-1)\beta$	$(c+a+2b-3)\alpha\beta\gamma-(c-2)\alpha\beta-(a-2)\beta\gamma-b\gamma\alpha-2\beta$
XY, YZ	$\frac{1}{2}[(\alpha-1)(\beta-1)\gamma+(\beta-1)(\gamma-1)\alpha]$	$(2a+b+2c-6)\alpha\beta\gamma-(a-2)\beta\gamma-(b-4)\gamma\alpha-(c-2)\alpha\beta-2\alpha-2\gamma$
YZ, ZX	$\frac{1}{2}[(\beta-1)(\gamma-1)\alpha+(\gamma-1)(\alpha-1)\beta]$	$(2b+c+2a-6)\alpha\beta\gamma-(b-2)\gamma\alpha-(c-4)\alpha\beta-(a-2)\beta\gamma-2\beta-2\alpha$
ZX, XY	$\frac{1}{2}[(\gamma-1)(\alpha-1)\beta+(\alpha-1)(\beta-1)\gamma]$	$(2c+a+2b-6)\alpha\beta\gamma-(c-2)\alpha\beta-(a-4)\beta\gamma-(b-2)\gamma\alpha-2\gamma-2\beta$
XY, YZ, ZX	$\frac{1}{2}(\alpha-1)(\beta-1)(\gamma-1)$	$\frac{1}{2}[(a+b+c-3)\alpha\beta\gamma-(a-2)\beta\gamma-(b-2)\gamma\alpha-(c-2)\alpha\beta-2\alpha-2\beta-2\gamma+3]$

groups of $c-1$ within the parallelopipeds whose ends lie in the planes $z=\pm 1$ and whose sides are planes (X) and (Y) for which $(t+t_1)/bc$ and $(t+t_2)/ca$ are integers, these double points also lying in planes (Z) for which $(t+t_3)/ab$ is an integer; (2°) $\frac{1}{2}(b-1)\gamma\alpha(\beta-1)$ points and (3°) $\frac{1}{2}(a-1)\beta\gamma(\alpha-1)$ points for which similar statements may be written down by mere rotation of letters.

14. Number of Curves. The question of the number of distinct curves we obtain by letting t_1, t_2, t_3 vary, presents some points of interest. Let us first find the relations that must hold between different sets of values of these in order that the geometrical loci determined may be the same. Suppose we have given a set of values of t_1, t_2, t_3. This determines a curve. Select arbitrarily some point t_0 on this curve. Let t_1', t_2', t_3' be a second set of constants and t_0' an arbitrary point on the curve determined by these. Then in order that the two curves may be geometrically the same it is necessary and sufficient that we be able to find Δt for the second equation such that

$$\Delta t \equiv [(t_0'+t_1')-(t_0+t_1)] \pmod{2bc\alpha},$$
$$\Delta t \equiv [(t_0'+t_2')-(t_0+t_2)] \pmod{2ca\beta},$$
$$\Delta t \equiv [(t_0'+t_3')-(t_0+t_3)] \pmod{2ab\gamma},$$

no matter what the original values of t_0 and t_0'. To do this it is necessary and sufficient that we be able to determine h, k, l so that

$$\Delta t = (t_0'+t_1')-(t_0+t_1)+2hbc\alpha = (t_0'+t_2')-(t_0+t_2)+2kca\beta$$
$$= (t_0'+t_3')-(t_0+t_3)+2lab\gamma,$$

that is,

$$t_1'-t_1+2hbc\alpha = t_2'-t_2+2kca\beta = t_3'-t_3+2lab\gamma.$$

The necessary and sufficient conditions that these last equations can be satisfied are that

(a) $\qquad (t_1-t_2) \equiv (t_1'-t_2') \pmod{2c}$,

(b) $\qquad (t_2-t_3) \equiv (t_2'-t_3') \pmod{2a}$,

(c) $\qquad (t_3-t_1) \equiv (t_3'-t_1') \pmod{2b}$.

It is worth noticing that these are not entirely independent. If any two, say (a) and (b), are satisfied by t_1, t_2, t_3 then it follows that, t_1', t_2', t_3' being fixed, $t_3-t_1 \equiv t_3'-t_1' \pmod 2$.

Suppose now we wish to learn how many curves of type A are possible. The above conditions that two points lie on the same curve show that it is only the relations between t_1, t_2, t_3 that count and that we may fix any one of

t_1, t_2, t_3 at pleasure. Suppose we fix t_3. For this type of curve none of the congruences $t_1 \equiv t_2 \pmod{c}$, $t_2 \equiv t_3 \pmod{a}$, $t_3 \equiv t_1 \pmod{b}$ can be satisfied. We can select an infinite number of values of t_1 and t_2 such that $(t_3 - t_1) < 2b$ and $(t_2 - t_3) < 2a$ and such that (b) and (c) are not satisfied. Only a finite number of pairs of such values of t_1 and t_2 can satisfy (a). Hence there must be an infinite number of curves of type A.

Next consider type B. Suppose $t_3 \equiv t_1 \pmod{b}$. We will again suppose t_3 fixed. Then there are only two possible values of t_1 less than $2b$ and not satisfying (c) (if the curve determined by t_1', t_2', t_3', be of type A and if the difference of no two of these be an integer, as we shall suppose from now on). But there are still an infinite number of values of t_2 not satisfying (b) nor the congruence $t_2 \equiv t_3 \pmod{a}$ and furthermore an infinite number of pairs of such values of t_1 and t_2 not satisfying (a) nor the congruence $t_1 \equiv t_2 \pmod{c}$. Hence there are an infinite number of curves of type B whose ZX projections are open curves. The same is true of curves whose XY or YZ projections are open curves.

But when we come to type C we find that the number of curves is finite. Suppose $t_2 \equiv t_3 \pmod{a}$, $t_3 \equiv t_1 \pmod{b}$. It follows from these that $t_1 \equiv t_2 \pmod{1}$. Suppose again we fix t_3. Then we have a series of values for t_1 that satisfy $t_3 \equiv t_1 \pmod{b}$ and do not satisfy (c). Similarly we have a series of values for t_2 that satisfy $t_2 \equiv t_3 \pmod{a}$ and do not satisfy (b). But among these values there are only $2(c-1)$ pairs not satisfying $t_1 \equiv t_2 \pmod{c}$ that make $t_1 - t_2$ less in absolute value than $2c$ and hence the number of curves whose YZ and ZX projections are open curves is $2(c-1)$. Similarly there are $2(a-1)$ curves whose ZX and XY projections are open curves, and $2(b-1)$ curves whose XY and YZ projections are open curves.

In case D we can get at the number of curves by a much simpler method. Each open curve, as we have seen, passes through two vertices of the cube and no two open curves can pass through the same vertex for if they did they would necessarily be tangent to one another at the vertex, a condition we can prove impossible by evaluating the indeterminate forms which the expressions for the slopes and their derivatives take on at a vertex. Furthermore we can determine an open curve such that it will pass through any required vertex. Since the cube has eight vertices there must be just four open curves.

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A SPECIAL QUADRI-QUADRIC TRANSFORMATION OF REAL POINTS IN A PLANE

By Carl C. Engberg

The transformation here studied is strictly speaking a (2, 2) transformation, for the equations of the direct transformation and of its inverse contain a double valued function, so that every point (x, y) goes over into two points (x', y'), and conversely.

By confining ourselves to real points of the plane, however, and making a suitable convention in regard to the sign of the radical, we reduce these double valued functions to single valued functions, and thus obtain a (1, 1) correspondence between the points of half the plane (viz., the quadrants above and below the lines $y = \pm x$) and the points of the whole plane.

For a general discussion of (2, 2) transformations the reader should consult articles by P. Visalli[*] and Burali-Forti.[†] The special transformation here studied has been employed by the writer in a paper on the Cartesian Oval,[‡] where many properties of these ovals are obtained by applying the transformation to a parabola.

1. *Definition of the Transformation.* The transformation here studied is defined by the equations

$$\left.\begin{aligned} x &= x' \\ y &= \pm\sqrt{x'^2 + y'^2} \end{aligned}\right\} \text{whence}: \left\{\begin{aligned} x' &= x \\ y' &= \pm\sqrt{y^2 - x^2}, \end{aligned}\right.$$

whereby we agree that the sign of y' shall be the same as the sign of y.

If the given point (x, y) is real, the transformed point (x', y') will be real or imaginary according as $x < y$ or $x > y$.

Geometrically speaking, the transformed point P' is that point whose abscissa is the abscissa of the given point P, and whose radius vector is the ordinate of P. If the abscissa of the given point is less than its ordinate, we can construct two such points P' but we agree to take that one which lies on the

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[*] *Rendiconti del Circolo Matematico di Palermo*, vol. 3 (1889), pp. 165-170.

[†] *Ibid.*, vol. 5 (1891), pp. 91-99.

[‡] *Graduate Bulletin of the University of Nebraska*, vol. 1 (1900), pp. 23-40.

same side of the axis of x as P does. If the abscissa of the given point is greater than its ordinate, the transformed point does not exist.

2. *Resulting Deformation of the Plane.* By this transformation a straight line through the origin, say $y = mx$, is carried over into another straight line through the origin, viz:

$$y' = \pm \sqrt{m^2 - 1}\, x',$$

where the sign of the radical is to be taken the same as the sign of m. When the inclination of the given line varies from $90°$ to $45°$ ($\infty > m > 1$), the inclination of the transformed line varies from $90°$ to $0°$; when it becomes less than $45°$ ($1 > m > 0$), the transformed line becomes imaginary.

The effect of the transformation may then be described as follows (confining ourselves to the upper half of the plane): the quadrantal region* above the lines $y = \pm x$ is expanded like a fan until its bounding lines coincide with the axis of x; the points of the axis of y remain fixed, while all other points of the region move towards the x-axis along lines perpendicular to that axis. The portion of the plane between the lines $y = \pm x$ and the x-axis becomes imaginary.

3. *Properties of the Transformation* (A).

If a curve is symmetrical with respect to the x-axis, its transformed curve will also be symmetrical with respect to that axis.

If two curves are tangent to each other at a given point, the corresponding curves will also be tangent to each other at the corresponding point.

The principal portion of a straight line $x = a$ perpendicular to the x-axis is transformed into the whole line $x = a$, the two points $(a, \pm a)$ uniting in the single point $(a, 0)$.

The principal portion of a straight line $y = b$ parallel to the x-axis goes over into the semicircle $x^2 + y^2 = b^2$; the pair of lines $y = \pm b$ gives the whole circle.

The principal portion of any straight line $y = mx + b$ is transformed into two quarters of an x-symmetric conic

$$y^2 = (m^2 - 1)x^2 + 2bmx + b^2,$$

* We shall speak of this region, together with the corresponding region in the lower half of the plane, as forming the "principal portion" of the plane; the "principal portion" of any curve shall then mean that portion of the curve which lies in the principal portion of the plane.

while the pair of lines $y = \pm (mx + b)$ gives the whole conic. The conic is tangent to the given lines where they cross the y-axis; one of its foci is at the origin; the corresponding directrix meets the given line on the x-axis; and the eccentricity is m. The conic will be an ellipse, parabola, or hyperbola according as the inclination of the given line is $< 45°$, $= 45°$, or $> 45°$.

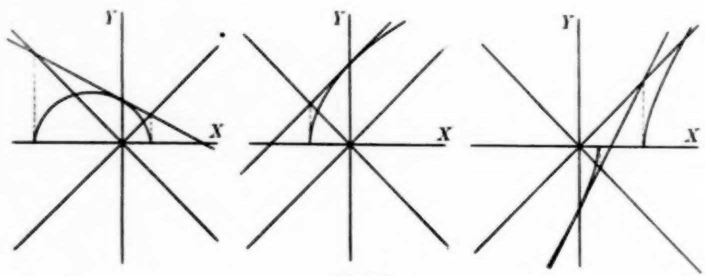

FIG. 1.

A set of parallel lines goes over into a set of x-symmetric conics having a constant eccentricity and a common focus at the origin.

The parabola $y^2 = 2ax$ goes over into the circle $x^2 + y^2 = 2ax$.

The parabola $y^2 - 2Ay - 2Bx + C^2 = 0$ goes over into the Cartesian Oval $\rho^2 - 2A\rho - 2Bx + C^2 = 0$.

An x-symmetric conic goes over into an x-symmetric conic, and if the centre is at the origin it remains there.

A quartic symmetric with respect to the x-axis goes over into another quartic having the same property.

The equilateral hyperbola $y^2 - x^2 + 2gx + 2fy + c = 0$ goes over into the quartic $y^2 + 2gx + 2f\rho + c = 0$, which in its turn goes over into a bicircular quartic.

4. *The Inverse Transformation* (B). The inverse transformation will clearly transform the whole plane into half the plane—viz. the half which lies above and below the lines $y = \pm x$. The following properties of the inverse transformation, which we shall designate as transformation B, will aid in giving a conception of it:

A straight line $y = mx + b$ is transformed by B into parts of an x-symmetric hyperbola

$$y^2 = (m^2 + 1)x^2 + 2bmx + b^2,$$

while the pair of lines $y = \pm (mx + b)$ gives the whole hyperbola. The hy-

perbola will lie wholly in the principal portion of the plane; it will be tangent to the lines $y = \pm x$, and also to the given lines. The slopes of its asymptotes will be $\pm \sqrt{m^2 + 1}$.

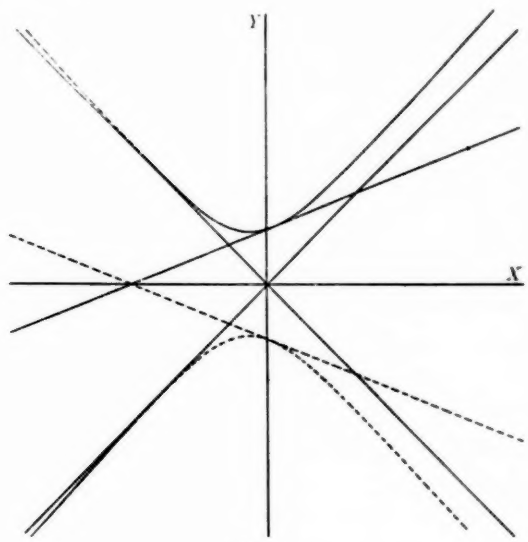

FIG. 2.

The circle $x^2 + y^2 = 2ax$ goes over into the parabola $y^2 = 2ax$.

The circle $(x - a)^2 + y^2 = r^2$ is transformed into the parabola

$$y^2 = 2ax + r^2 - a^2.$$

In general, B transforms an x-symmetric conic into another x-symmetric conic, and if the centre is at O it will remain there. If the focus of the conic is at O, the conic goes over into a pair of straight lines.

The limaçon $\rho = 2a \cos\theta \pm c$ is transformed into two equal parabolas

$$(y \pm c/2)^2 = 2ax + c^2/4.$$

The conchoid $a = (\rho \pm c)\cos\theta$ is transformed into two equal equilateral hyperbolas $xy = ay \pm cx$.

The cissoid $\rho = 2a(\sec\theta - \cos\theta)$ goes over into the cubic

$$xy^2 + 2a(x^2 - y^2) = 0.$$

5. *Applications.* We now proceed to derive a few theorems from

known theorems on conics and straight lines by means of the direct transformation (A) or its inverse (B).

1. "A variable line through O cuts the line $x = a$ in P, and Q is taken on this variable line so that $OP \times OQ$ is constant: the locus of Q is a circle through O."

Transforming by (B) we have:

A variable line through O cuts the line $x = a$ in P, and Q is taken on this variable line so that the product of the ordinates of P and Q is constant; the locus of Q is a parabola through O.

2. "A variable line through O cuts the circle $(x - a)^2 + y^2 = p^2$ in P, and Q is taken on this line so that $OP \times OQ$ (or $OP \div OQ$) is constant; the locus of Q is an x-symmetric circle."

Transforming by (B), we have:

A variable line through O cuts an x-symmetric parabola in P, and Q is taken on this line so that the product (or the quotient) of the ordinates of P and Q is constant; the locus of Q is another x-symmetric parabola.

3. "A variable line which moves always parallel to itself cuts two fixed lines in P and Q; the locus of the middle point of PQ is a straight line through the point of intersection of the two fixed lines."

Transforming by (A) we have:

A variable x-symmetric conic, having a focus at O and a constant eccentricity, meets two fixed x-symmetric conics having a common focus at O in the points P and Q on the same side of the axis. On the variable conic a point is taken whose abscissa is the arithmetic mean of the abscissas of P and Q; then the locus of this point is another x-symmetric conic having a focus at O and passing through the points of intersection of the two fixed conics.

Transforming by (B) we have another theorem obtained from this by changing the word "conic" to "hyperbola," and the words "having a focus at O" to the words "tangent to the lines $y = \pm x$."

4. "A variable line through O meets the $\left\{\begin{array}{l}\text{circle } x^2 + y^2 = 2ax \\ \text{line } x = a\end{array}\right\}$ in P, and Q is taken on this variable line at a constant distance $\pm c$ from P. Then the locus of Q is the $\left\{\begin{array}{ll}\text{limaçon} & \rho = 2a\cos\theta \pm c \\ \text{conchoid} & a = (\rho + c)\cos\theta\end{array}\right\}$."

Transforming by (B) we have:

A variable line through O meets the $\left\{\begin{array}{l}\text{parabola } y^2 = 2ax \\ \text{line } x = a\end{array}\right\}$ in P, and

Q is taken on this variable line so that the difference between the ordinates of P and Q is a constant, $\pm c$. Then the locus of Q is two equal $\left\{\begin{matrix}\text{parabolas} \\ \text{equilateral hyperbolas}\end{matrix}\right\}$ through O.

5. "In the parabola $y^2 = 4ax$ let P, Q, R be points whose ordinates are in geometric progression: then the tangents at P and R meet on the ordinate of Q."

Transforming by (A) we have:

In the circle $x^2 + y^2 = 4ax$ let P, Q, R be three points whose radii vectores are in geometric progression; then the two x-symmetric conics having a common focus at O and touching the circle, the one at P and the other at R, will meet on the ordinate of Q.

6. "A variable line through O meets the fixed circle $x^2 + y^2 = 2ax$ in P and the fixed tangent $x = 2a$ in Q. On this variable line take $OR = PQ$; then the locus of R is the cissoid $\rho = 2a(\sec\theta - \cos\theta)$."

Transforming by (A) we have:

A variable line through O meets the fixed parabola $y^2 = 2ax$ in P and the fixed line $x = 2a$ in Q. On this variable line the point R is taken whose ordinate is the difference between the ordinates of P and Q; then the locus of R is the cubic $xy^2 + 2a(x^2 - y^2) = 0$.

In conclusion we may note that the transformation may be easily extended to three dimensions, the equations of the transformation being

$$x = x', \quad y = y', \quad z = \pm\sqrt{x'^2 + y'^2 + z'^2}.$$

Here the sign of the radical is to be taken the same as the sign of z; the "principal portion" of space will then be the region above the four planes $z = \pm x$, $z = \pm y$, together with the corresponding region below the xy-plane.

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CONTENTS

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